Spectra of Conformal Field Theories and Hyperbolic Manifolds

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What Is Quantum Field Theory?

1. How do we rigorously define quantum field theory?

2. How do we compute observables?

Situation better for **conformal field theories**:

- Precise axiomatic formulation in any number of dimensions.
- Effective for computations, even leading to new predictions.

} conformal bootstrap

CFT Axioms

1. V = a unitary representation of the conformal group in d dimensions.

   - V = space of states = space of local operators.
   - Decompose into irreducible representations: $V = \bigoplus_i V_{\Delta_i,\rho_i}$.
   - Local operators: $\mathcal{O}_i(x)$ with $x \in B^d$ generates V_{Δ_i,ρ_i}.

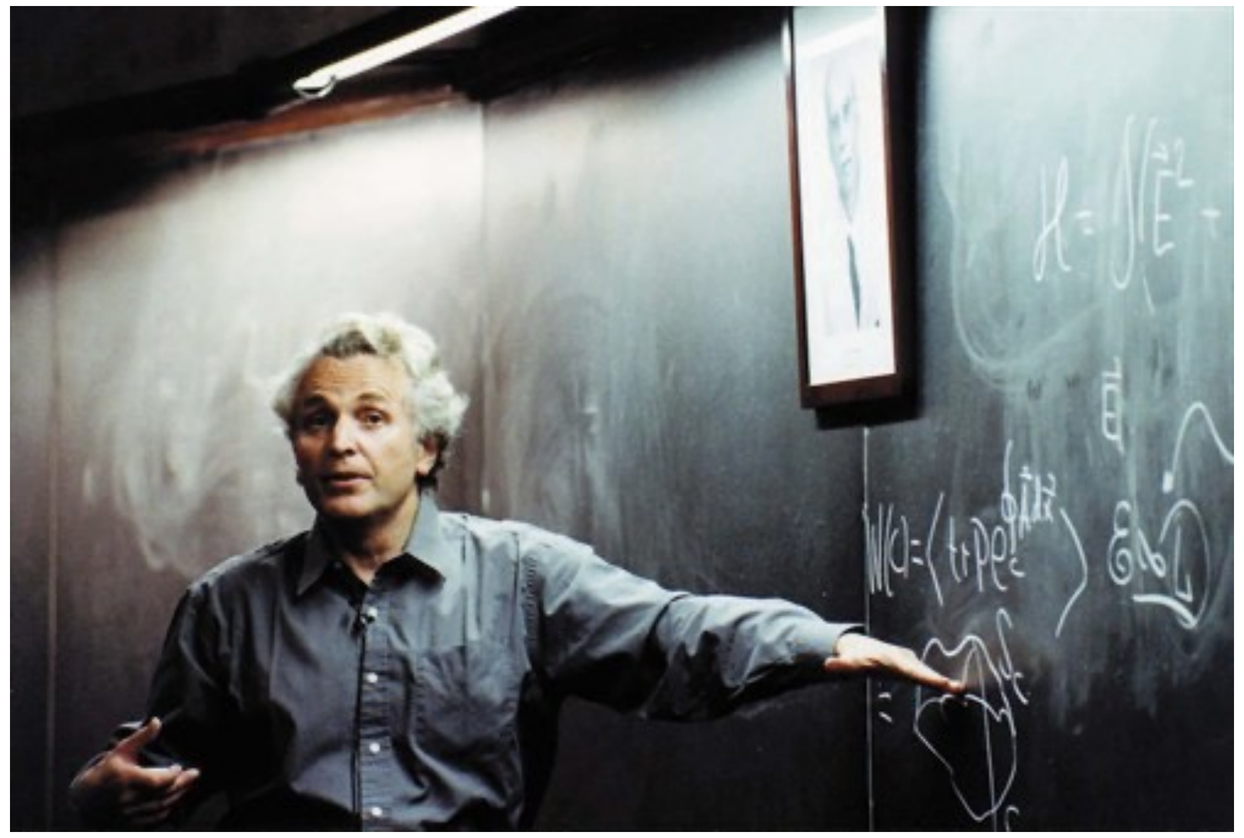
2. Operator product expansion: $\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k c_{ijk} |x-y|^{-\Delta_i-\Delta_j+\Delta_k} \mathcal{O}_k(y)$.

3. Associativity: $\mathcal{O}_i(x)(\mathcal{O}_j(y)\mathcal{O}_k(z)) = (\mathcal{O}_i(x)\mathcal{O}_j(y))\mathcal{O}_k(z)$

$\Rightarrow$ stringent constraints on the spectrum Δ_i, ρ_i and structure constants c_{ijk} .

Long term goal: Solve and classify CFTs in general dimension starting from these axioms.

A. Polyakov: "I was dreaming in the 1970s to have a classification of fixed points based on the operator product expansion. The program was successful in two dimensions, and I think it is not excluded that in three dimensions something like that is still possible."



Current status:

- $d = 2$: partial progress (rational theories, Liouville theory).
- $d > 2$: The only solved examples are free theories, but infinitely many interacting examples surely exist.

The conformal bootstrap: hopes and challenges

Hopes

- The conformal bootstrap axioms are complete = all solutions arise from physical conformal field theories.
- The only solutions with $d > 6$ are free fields.
- Generic (local) solutions with $d > 2$ are rigid = admit no continuous deformations.

Challenges

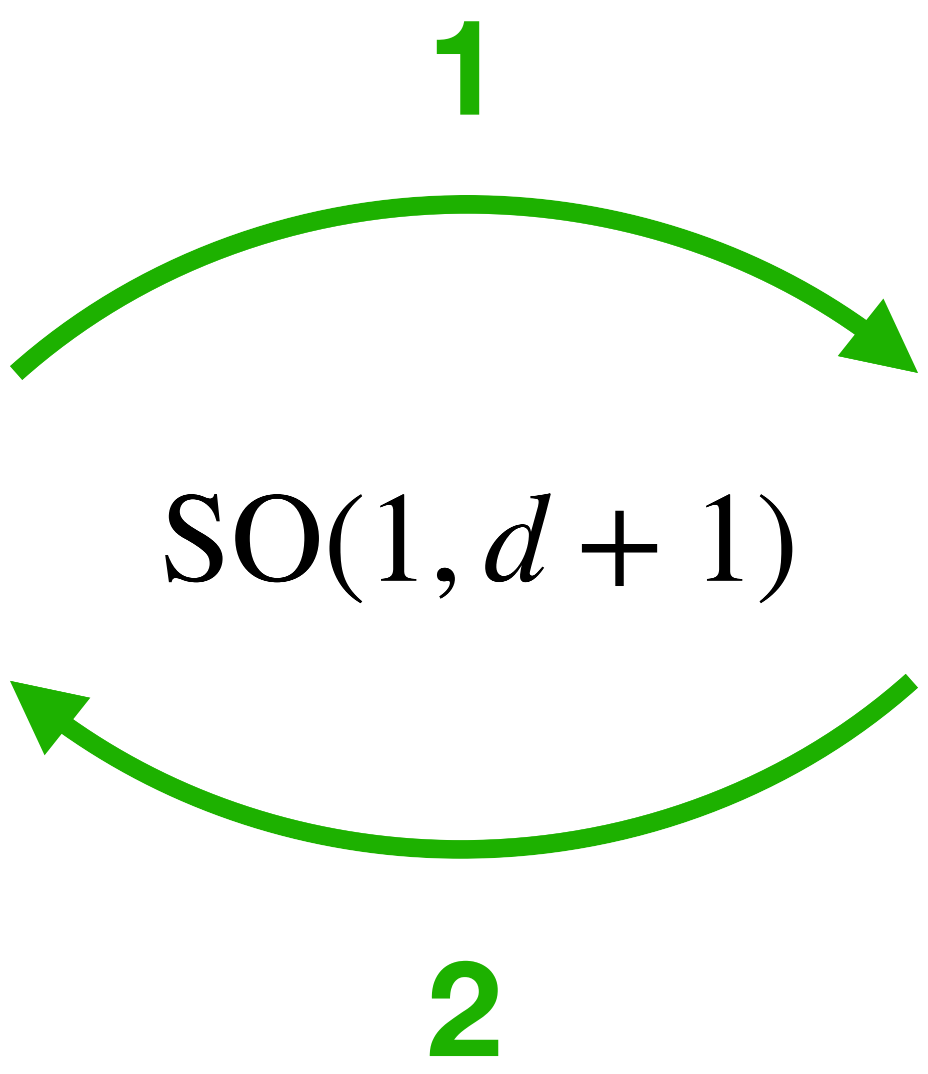
- No explicitly constructed solutions besides free fields for $d > 2$.
- Generic conformal field theory expected to exhibit chaos (level repulsion).

This talk

- Solutions of the full conformal bootstrap (in any d) from hyperbolic manifolds.
- This provides solutions of the conformal bootstrap with chaotic spectra.

This talk

conformal field theory
in d dimensions



spectral geometry
of hyperbolic $(d+1)$-manifolds

1. Rigorous estimates on spectra of hyperbolic manifolds from conformal field theory.

[Bonifacio, Hinterbichler ’19+’20], [Bonifacio, ’21+’21]
[Kravchuk, DM, Pal, ’21], [Bonifacio, DM, Pal ’23]
[Adve, Bonifacio, DM, Pal, Sarnak, Xu WIP]

2. A novel viewpoint on conformal field theory in general dimension.

[Bonifacio, Kravchuk, DM, Pal WIP]

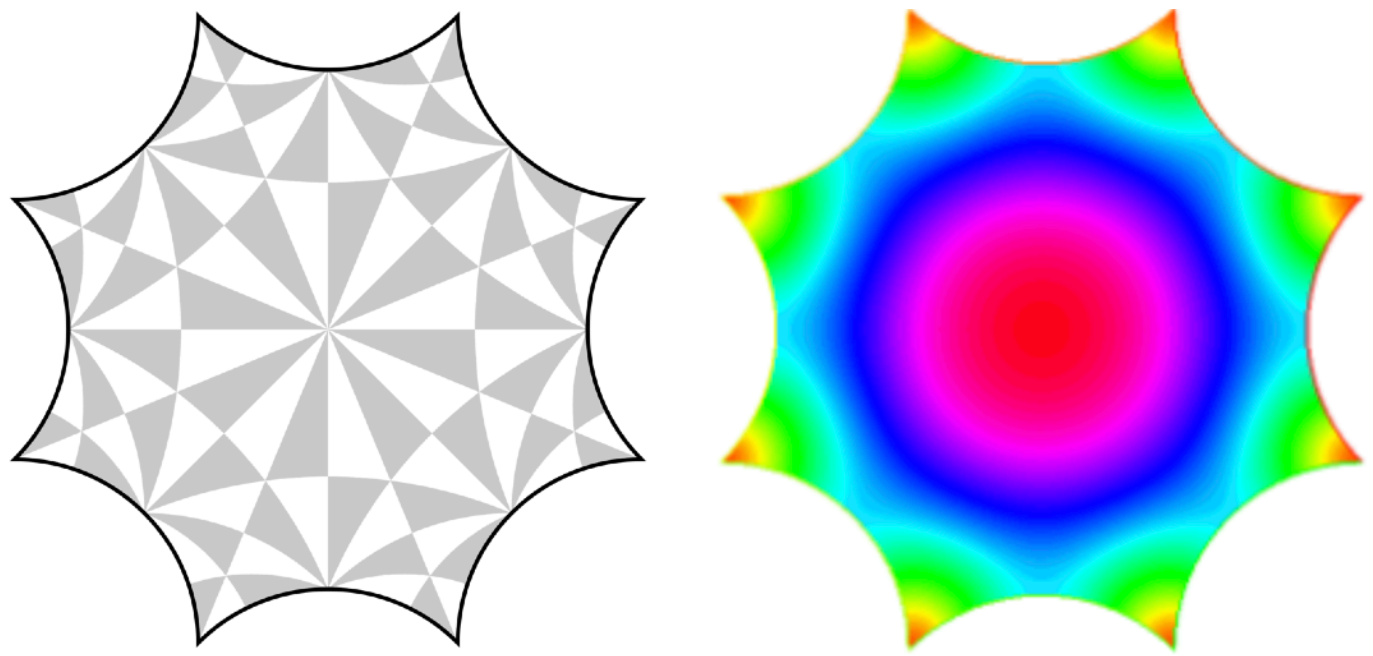
Spectra of hyperbolic manifolds

- Hyperbolic manifold M = Riemannian manifold of constant sectional curvature -1.
- Equivalently $M = \Gamma \backslash \mathbb{H}^d$, where $\mathbb{H}^d = \mathrm{SO}(d,1)/\mathrm{SO}(d)$ and Γ is a discrete subgroup of $\mathrm{SO}(d,1)$.

- Given M, consider the Laplace equation

$$\Delta_M h_i = \lambda_i h_i$$

with spectrum of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$.



- The high-energy spectrum typically exhibits quantum chaos.
- Focus on the spectral gap λ_1.

Conjecture (Selberg): If $d = 2$ and Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then $\lambda_1 \geq 1/4$.

Question: What values does λ_1 assume as Γ ranges over all cocompact subgroups of $\mathrm{SO}(d,1)$?

Today: Answer this question for $d = 2$ by adapting the conformal bootstrap to hyperbolic manifolds.

Unifying conformal field theories and hyperbolic manifolds

[Bonifacio, Kravchuk, DM, Pal WIP]

Main idea: Define a mathematical object describing both conformal field theories and hyperbolic manifolds.

Recall that a measure space $(Y, \mathscr{A}, \mu)$ consists of an underlying set Y, a σ-algebra of measurable sets $\mathscr{A}$, and a measure function $\mu : \mathscr{A} \to \mathbb{R}_{\geq 0}$.

Definition: A *conformal measure space* is a measure space $(Y, \mathscr{A}, \mu)$ with an action of $G = \mathrm{SO}(d,1)$.

Example 1 *(d-dimensional conformal field theory)*:

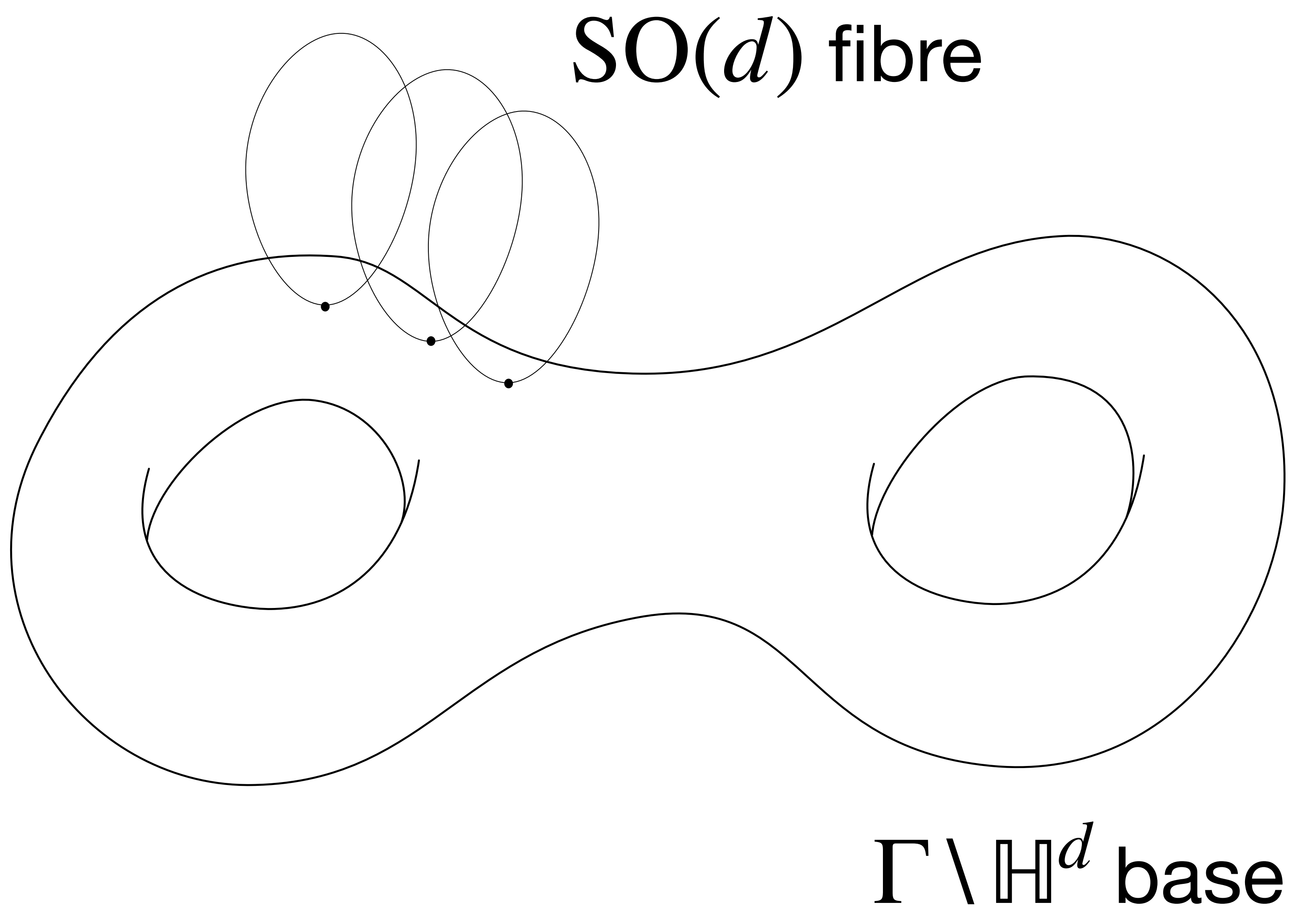
- The path integral for quantum field theory on a manifold X = measure on a space of distributions $\mathscr{S}'(X)$.
- d-dimensional conformal field theory on S^d = measure on $\mathscr{S}'(S^d)$ invariant under $\mathrm{SO}(d+1,1)$.
- Hard to construct explicitly, here taken as a definition of CFT.

Example 2 *(hyperbolic d-manifolds)*:

- Let $M = \Gamma \backslash \mathbb{H}^d$ be a hyperbolic d-manifold, for some $\Gamma \subset \mathrm{SO}(d,1)$.
- Then $Y = \Gamma \backslash \mathrm{SO}(d,1)$ is a conformal measure space, with μ = Haar measure on $\mathrm{SO}(d,1)$.
- $\mathrm{SO}(d,1)$ acts on Y by right translations.

Rest of the talk: Formulate the conformal bootstrap as a method to study conformal measure spaces.

Visualizing $\Gamma \backslash \mathrm{SO}(d,1)$



$$\mathrm{SO}^0(d,1) \to \Gamma \backslash \mathrm{SO}^0(d,1) \to \Gamma \backslash \mathbb{H}^d$$

$$\mathrm{SO}^0(d,1) \to \mathbb{H}^d = \mathrm{SO}^0(d,1) \,/\, \mathrm{SO}(d) \to \Gamma \backslash \mathbb{H}^d$$

The spectrum of a conformal measure space

- Fix a conformal measure space (Y, μ) and consider the Hilbert space of random variables $L^2(Y, \mu)$.
- Since $G = \mathrm{SO}(d+1,1)$ acts on (Y, μ), $L^2(Y, \mu)$ is a unitary representation of G.
- When (Y, μ) is sufficiently nice, $L^2(Y, \mu)$ decomposes into unitary irreducible representations of G.

The unitary dual of $\mathrm{SO}(d+1,1)$ classified by Bargmann, Gelfand, Naimark, Thomas, Dixmier, Hirai, Takahashi, Thieleker

- Besides the trivial representation, all unitary irreps of G are infinite-dimensional.
- Representation $R_{\Delta,\rho}$ labelled by $\Delta \in \mathbb{C}$ and $\rho \in \widehat{\mathrm{SO}(d)}$={ Young diagrams with $\leq d/2$ rows }.
- $R_{\Delta,\rho}$ realized as a space of functions $\mathbb{R}^d \to \rho$.

1. Trivial representation $R_{0,0} \simeq \mathbb{C}$.
2. Principal series $R_{\Delta,\rho}$ with $\Delta \in d/2 + i\mathbb{R}$.
3. Complementary series $R_{\Delta,\rho}$ with $\Delta \in (m, d-m)$, where m = # of rows of ρ.
4. Exceptional series $R_{\Delta,\rho}$, discrete values of Δ for fixed ρ.

Definition (*spectrum of a conformal measure space*):

The *spectrum* of (Y, μ) is the set of (Δ_i, ρ_i) appearing in $L^2(Y, \mu) \simeq \bigoplus_{i=0}^{\infty} R_{\Delta_i,\rho_i}$.

Interpreting the spectrum

Example 1 *(d-dimensional conformal field theory)*:

- Let (Y, μ) be a c.m.s. with Y = space of distributions $\phi(x)$ on S^d, μ = path integral measure.
- Suppose $\phi(x)$ transforms like a conformal field of scaling exponent $\Delta \in (0, d/2)$.
- Claim: $L^2(Y, \mu)$ contains the complementary series irrep $R_{\Delta,0}$ of $\mathrm{SO}(d+1,1)$.

Embedding $R_{\Delta,0} \to L^2(Y, \mu)$ provided by $f \mapsto (f, \phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx$.

More general observables $\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} f(x_1, \dots, x_k)\phi(x_1)\dots\phi(x_k)dx_1\dots dx_k$ lead to a continuous spectrum.

Conclusion: Spectrum of (Y, μ) related to the spectrum of scaling exponents.

- L^2(2d Ising) $\simeq R_{0,0} \oplus R_{1/8,0} \oplus$ continuous spectrum
- L^2(3d Ising) $\simeq R_{0,0} \oplus R_{0.5181\dots,0} \oplus R_{1.413\dots,0} \oplus$ continuous spectrum

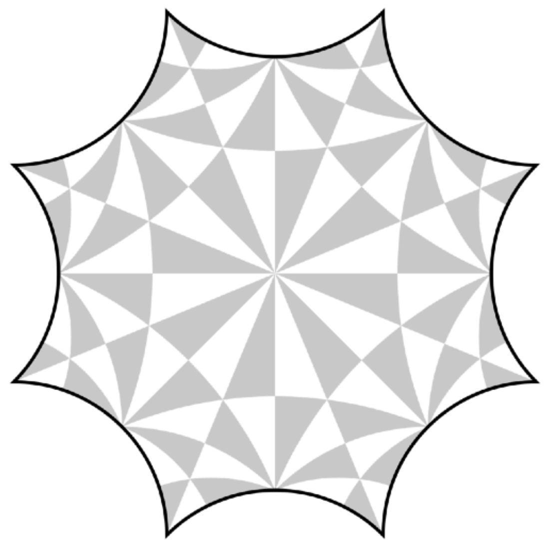
Interpreting the spectrum

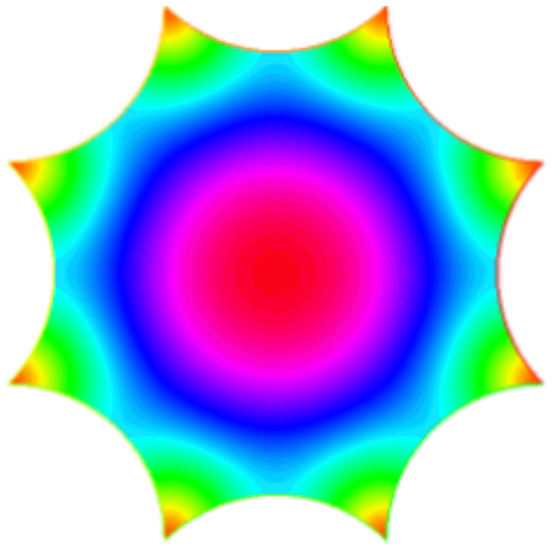
Example 2 *(hyperbolic d-manifolds)*:

- $R_{\Delta,0}$ appears as a direct summand in $L^2(\Gamma \backslash \mathrm{SO}(d,1))$ ⇔ The Laplace operator on $\Gamma \backslash \mathbb{H}^d$ has an eigenvalue $\lambda = \Delta(d-1-\Delta)$.
- More generally $R_{\Delta,\rho} \subset L^2(\Gamma \backslash \mathrm{SO}(d,1))$ are in one-to-one correspondence with solutions of the Laplace equation on sections of the vector bundle over $\Gamma \backslash \mathbb{H}^d$ associated with ρ.

$\Rightarrow L^2(\Gamma \backslash \mathrm{SO}(d,1)) \simeq \bigoplus_{i=0}^{\infty} R_{\Delta_i,\rho_i}$ restricted to the subspace of $\mathrm{SO}(d)$-invariant vectors equivalent to the spectral theorem for the Laplacian on $\Gamma \backslash \mathbb{H}^d$.

Conclusion: Spectrum of $\Gamma \backslash \mathrm{SO}(d,1)$ ⇔ spectrum of Laplace operators on $\Gamma \backslash \mathbb{H}^d$.

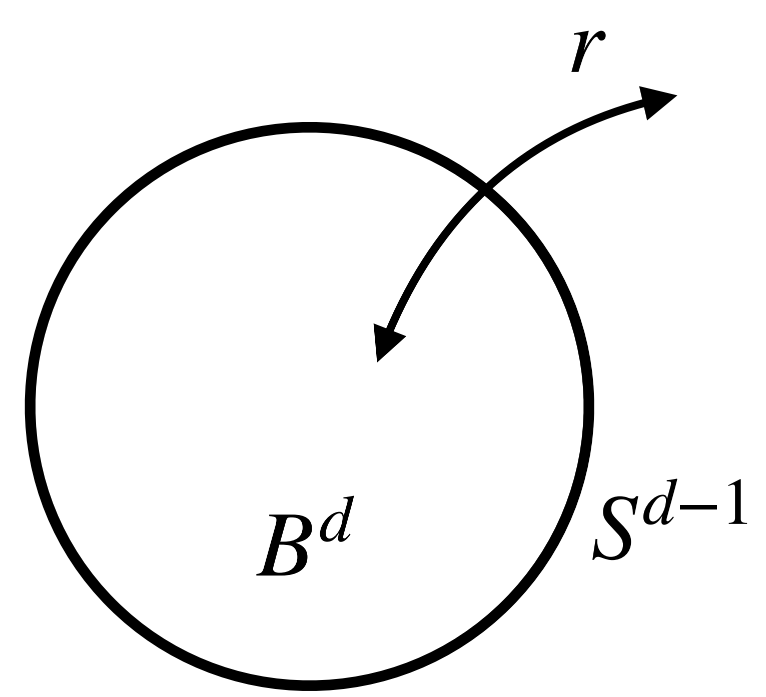




Space of observables vs. space of states

We have considered $L^2(Y, \mu)$ in conformal field theory

- A typical element of $L^2(Y, \mu)$: $F[\phi] = \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} f(x_1, \dots, x_k)\phi(x_1)\dots\phi(x_k)dx_1\dots dx_k$
- $L^2(Y, \mu)$ is a unitary representation of the **Euclidean conformal group** $\mathrm{SO}(d+1,1)$.
- Inner product: $(F_1, F_2) := \langle \overline{F_1} F_2 \rangle$.



How to recover the physical Hilbert space on S^{d-1} (spanned by local operators):

- Restrict $L^2(Y, \mu)$ to V := observables supported in B^d.
- Introduce a twisted inner product on V: $(F_1, F_2)_L := \langle \overline{r(F_1)} F_2 \rangle$, r = the sphere inversion.
- Reflection positivity $\Leftrightarrow$ unitarity $\Leftrightarrow$ $(F, F)_L \geq 0$.
- Mod out by null states and complete to get the physical Hilbert space W.
- W is a unitary representation of the **Lorentzian conformal group** $\widetilde{\mathrm{SO}}(d,2)$.

Correlations

- Consider a conformal measure space (Y, μ) with spectral decomposition $L^2(Y, \mu) \simeq \bigoplus_{i=0}^{\infty} R_{\Delta_i, \rho_i}$.
- Introduce G-equivariant maps $\mathcal{O}_i : R_{\Delta_i, \rho_i} \to L^2(Y, \mu)$.
- Main quantities of interest: correlations of products $\langle \mathcal{O}_{i_1}(f_1)\ldots\mathcal{O}_{i_m}(f_m)\rangle := \int_Y \mathcal{O}_{i_1}(f_1)\ldots\mathcal{O}_{i_m}(f_m)\, d\mu(y)$.

Interpretation:

1. When (Y, μ) arises from a CFT and $\mathcal{O}(f) = \int_{\mathbb{R}^d} f(x)\phi(x)dx$, get m-point correlations of ϕ.
2. When (Y, μ) arises from a hyperbolic manifold, get integrals of products of Laplace eigenfunctions.

$$\langle \mathcal{O}_i(f)\rangle = 0$$

$$\langle \mathcal{O}_i(f_1)\mathcal{O}_j(f_2)\rangle = \delta_{ij}\int\int |x_1 - x_2|^{-2\Delta_i} f_1(x_1)f_2(x_2)dx_1dx_2$$

$$\langle \mathcal{O}_i(f_1)\mathcal{O}_j(f_2)\mathcal{O}_k(f_3)\rangle = c_{ijk}\int\int\int |x_1 - x_2|^{-\Delta_i-\Delta_j+\Delta_k}|x_1 - x_3|^{-\Delta_i-\Delta_k+\Delta_j}|x_2 - x_3|^{-\Delta_j-\Delta_k+\Delta_i} f_1(x_1)f_2(x_2)f_3(x_3)dx_1dx_2dx_3$$

- Hyperbolic manifold: $c_{ijk} = \int_{\Gamma\backslash\mathbb{H}^d} h_i(y)h_j(y)h_k(y)d\mu(y)$ for Laplace eigenfunctions h_i, h_j, h_k.
- For arithmetic hyperbolic manifolds, c_{ijk} related to values of L-functions.

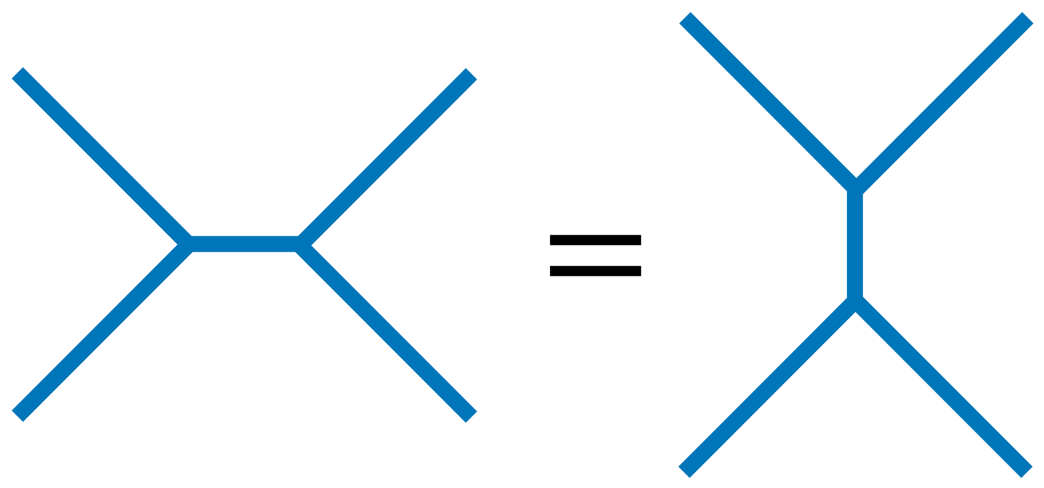
Product expansion and the conformal bootstrap

- Identify a subalgebra $\mathscr{A} \subset L^2(Y,\mu)$ and let $\mathcal{O}_i(f_1), \mathcal{O}_j(f_2) \in \mathscr{A}$.

$$\mathcal{O}_i(f_1)\mathcal{O}_j(f_2) = \sum_{k=0}^{\infty} c_{ijk}\,\mathcal{O}_k(f_1 \star f_2) \qquad \text{(product expansion)}$$

- Apply the product expansion to $\langle \mathcal{O}_i(f_1)\mathcal{O}_j(f_2)\mathcal{O}_k(f_3)\mathcal{O}_\ell(f_4)\rangle$ to get the spectral identities

$$\sum_{m=0}^{\infty} c_{ijm}c_{k\ell m}\,\Psi_m^{ijk\ell}(f_1,f_2,f_3,f_4) = \sum_{m=0}^{\infty} c_{i\ell m}c_{jkm}\,\Psi_m^{i\ell kj}(f_1,f_4,f_3,f_2)$$



(conformal bootstrap)

- Here $\Psi_m^{ijk\ell}$ are the <u>conformal partial waves</u>, fixed by representation theory of $\mathrm{SO}(d+1,1)$.

Conclusion: The spectral data $((\Delta_i,\rho_i))_{i=0}^{\infty}$, c_{ijk} of any conformal measure space must satisfy the conformal bootstrap equations.

Linear programming bounds

Idea

- Turn the spectral identities into estimates on the spectrum using linear programming.
- Consider the set of identities arising from $i = j = k = \ell$:

$$\sum_{m=0}^{\infty} (c_{iim})^2 \underbrace{\left[\Psi_m^{iiii}(f_1, f_2, f_3, f_4) - \Psi_m^{iiii}(f_1, f_4, f_3, f_2)\right]}_{A} = 0$$

Ingredients

1. $c_{iim} \in \mathbb{R} \Rightarrow (c_{iim})^2 \geq 0.$
2. Choose f_1, f_2, f_3, f_4 such that $A \geq 0$ whenever $(\Delta_m, \rho_m) \notin U$ for some $U \subset \widehat{G}$.

It follows that the spectrum of every conformal measure space must have a nonempty intersection with U.

Estimates on the spectral gaps of hyperbolic surfaces

Theorem [Kravchuk, DM, Pal '21], [Bonifacio '21]

1. Every hyperbolic orbifold of genus two satisfies: $\lambda_1 \leq 3.8388977.$

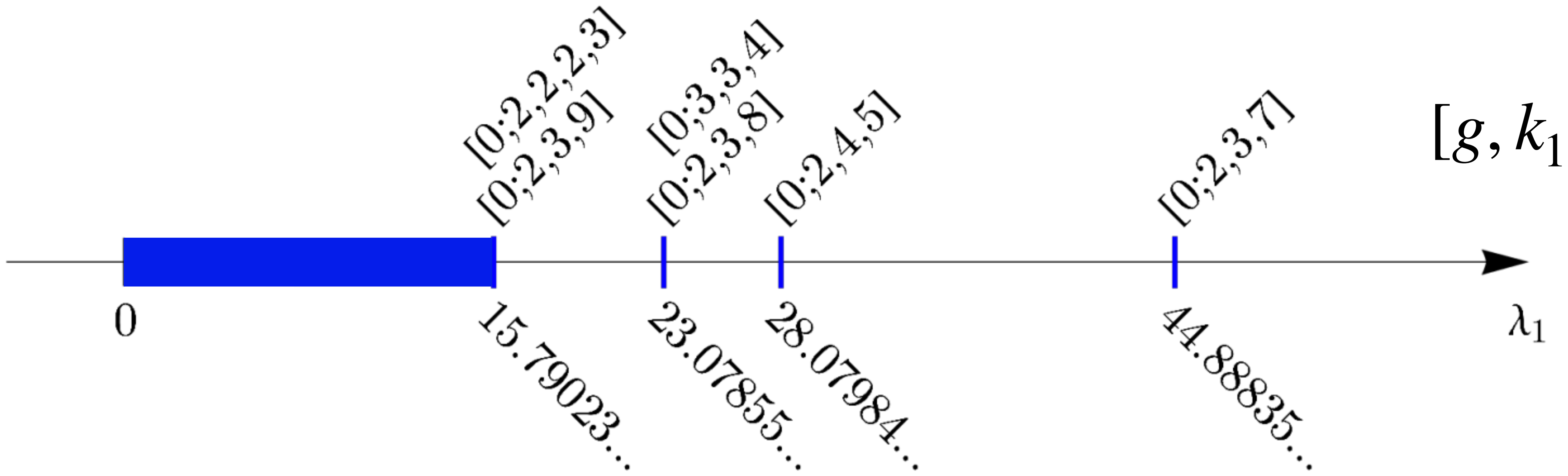
   Bolza surface: $\lambda_1 \approx 3.83887258$

2. Every hyperbolic orbifold of genus three satisfies: $\lambda_1 \leq 2.6784824.$

   Klein quartic: $\lambda_1 \approx 2.6779$

Question: What values does $\lambda_1(\Gamma)$ assume as Γ ranges over <u>all</u> cocompact subgroups of $\mathrm{PSL}_2(\mathbb{R})$?

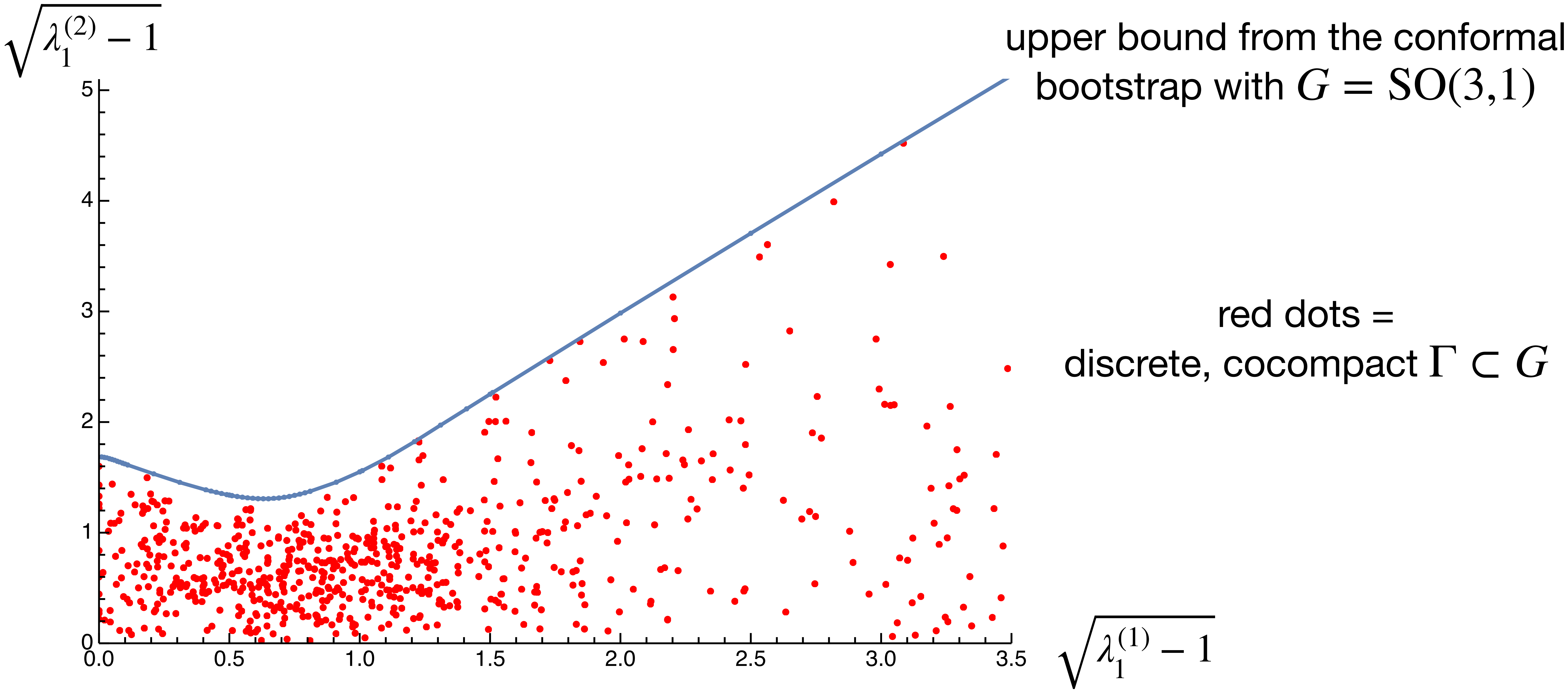
Answer: [Kravchuk, DM, Pal '21], [Adve, Bonifacio, DM, Pal, Sarnak, Xu WIP]



$[g, k_1, \ldots, k_r]$: genus g, elliptic points of orders $k_1, \ldots, k_r$

Estimates on the spectral gaps of hyperbolic 3-manifolds

[Bonifacio, DM, Pal ’23]



$\lambda_1^{(J)}$ = (spectral gap of the Laplacian acting on symmetric traceless tensors of rank J).

Natural question:

- Why is the conformal bootstrap method so powerful?
- Is it in some sense complete?

implying, in principle, any true statement about the spectral data of $\Gamma \backslash G$,

Theorem (Adve, to appear): Yes, at least for cocompact subgroups of $G = \mathrm{PSL}_2(\mathbb{R})$.

More precise statement: Every putative discrete spectrum with at most polynomial growth which solves all the conformal bootstrap identities is in fact the spectrum of $\Gamma \backslash \mathrm{PSL}_2(\mathbb{R})$ for some Γ.

Summary

- Hyperbolic d-manifolds and conformal field theories in $d-1$ dimensions described by the same type of mathematical objects.
- These objects are measure spaces with an action of the group $\mathrm{SO}(d,1)$.
- Associativity of multiplication gives a large set of identities for the spectral data of such measure spaces.
- These identities and linear programming produce (nearly) sharp spectral bounds.

Future directions

- In the context of conformal field theories, our formulation does not rely on unitarity. The Hilbert space of states replaced by the Hilbert space of observables.
- Can we use it to prove new spectral estimates in non-unitary CFTs, e.g. percolation?
- Finite-volume hyperbolic d-manifolds rigid for $d > 2$ (Mostow). Is this related to rigidity of higher-dimensional conformal field theories?
- Improve bounds on triple products c_{ijk} as $\lambda_k \to \infty$ towards the Lindelöf hypothesis for L-functions.

Thank you!